Advanced Robot Control
Static linearisation

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Presentation compiled for taking notes during lecture

Outline

Types of description (1/4)

The mobile robot can be described using different types of description [1]:

- In generalized coordinates,
- In auxiliary velocities,
- In linearised coordinates.

Types of description (2/4)

Generalized coordinates

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D(q) = A^T\lambda + Bu \tag{1}$$

Types of description (3/4)

Auxiliary velocities

$$\begin{cases} M^*\dot{\eta} + C^*\eta + D^* = B^*u, \\ \dot{q} = G\eta. \end{cases} \tag{2}$$

$$\eta, u \in R^m \tag{3}$$

Types of description (4/4)

Linearising coordinates

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{4}$$

Dynamics (1/2)

For a robotic system a Lagrangian can be defined as a difference between kinetic energy and potential energy

$$L(q, \dot{q}) = K(q, \dot{q}) - V(q). \tag{5}$$

From Hamilton's principle of least action the dynamics can be expressed as Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{6}$$

If there are external forces (friction, control forces, constraints) then above can be rewritten as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F \tag{7}$$

where F is generalized non-potential forces.

Dynamics (2/2)

If constraints forces are being considered as well as control forces then ultimately the dynamics can be expressed as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = A^T(q)\lambda + B(q)u. \tag{8}$$

Finally, the dynamics can be presented as

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D(q) = A^T\lambda + Bu \tag{9}$$

Kinematics

Let us consider following unicycle kinematics

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\Phi}_1 \\ \dot{\Phi}_2 \end{pmatrix} = \begin{bmatrix} \cos\theta & \cos\theta \\ \sin\theta & \sin\theta \\ \frac{1}{L} & -\frac{1}{L} \\ \frac{2}{R} & 0 \\ 0 & \frac{2}{R} \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \tag{10}$$

where $\eta_1 = \frac{R}{2}\dot{\Phi}_1$ and $\eta_2 = \frac{R}{2}\dot{\Phi}_2$ are scaled wheels' speeds. R is wheel radius and L is half distance between wheels.

Dynamics (1/3)

Let us consider dynamics given as

$$M(q)\ddot{q} = A^T\lambda + Bu \tag{11}$$

then

$$M = \begin{bmatrix} m_p & 0 & 0 & 0 & 0 \\ 0 & m_p & 0 & 0 & 0 \\ 0 & 0 & I_p & 0 & 0 \\ 0 & 0 & 0 & I_w & 0 \\ 0 & 0 & 0 & 0 & I_w \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{12}$$

Dynamics (2/3)

Given that kinematics of a mobile platform is given as

$$\dot{q} = G\eta \tag{13}$$

and

$$\ddot{q} = \dot{G}\eta + G\dot{\eta} \tag{14}$$

then the dynamic expressed in generalized coordinates can be rewritten using auxiliary velocities.

$$M(\dot{G}\eta + G\dot{\eta}) = A^T\lambda + Bu \tag{15}$$

yielding

$$MG\dot{\eta} + M\dot{G}\eta = A^T\lambda + Bu. \tag{16}$$

Dynamics (3/3)

The above can be left-handed multiplied by G^T

$$G^T MG\dot{\eta} + G^T M\dot{G}\eta = G^T Bu. \tag{17}$$

The equivalent form is

$$M^*\dot{\eta} + C^*\eta = B^*u \tag{18}$$

where $M^* = G^T MG$, $C^* = G^T M\dot{G}$ and $B^* = G^T B$.
The dynamics expressed in auxiliary velocities is an entry-point for static linearisation.

Linearised coordinates

- ξ_1 – group of coordinates that can be linearised statically,
- ξ_2 – group of coordinates that **can not** be linearised statically.

Let

$$\begin{aligned} \xi_1 &= h(q) \in R^m, \\ \xi_2 &= k(q) \in R^{n-m}. \end{aligned}$$

Differentiation of ξ_1

$$\xi_1 = h(q) \tag{19}$$

$$\dot{\xi}_1 = \frac{\partial h}{\partial q}\frac{dq}{dt} = \frac{\partial h}{\partial q}\dot{q} = \frac{\partial h}{\partial q}G\eta = R^{-1}\eta \tag{20}$$

The matrix R^{-1} is defined as

$$R^{-1} = \left[\frac{\partial h}{\partial q}G\right]_{m\times m} \tag{21}$$

The matrix R^{-1} is called a **decoupling matrix.**

Regularity condition

$$detR^{-1} \neq 0 \tag{22}$$

Let's assume that the above condition is fulfilled then

$$\dot{\xi}_1 = R^{-1}\eta$$

after left-hand side multiplication by R matrix we get

$$R/ \ \dot{\xi}_1 = R^{-1}\eta$$

$$\Leftrightarrow$$

$$\eta = R\dot{\xi}_1, \tag{23}$$

$$\dot{\eta} = \dot{R}\dot{\xi}_1 + R\ddot{\xi}_1. \tag{24}$$

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Reformulation of dynamic equations (1/3)

After substitution of (23)–(24) to (2) we get

$$M^* \left(\dot{R}\dot{\xi}_1 + R\ddot{\xi}_1 \right) + C^* R\dot{\xi}_1 + D^* = B^* u. \tag{25}$$

After left-hand side multiplication by the transposition of R matrix we get

$$R^T / \;\; M^* R\ddot{\xi}_1 + M^* \dot{R}\dot{\xi}_1 + C^* R\dot{\xi}_1 + D^* = B^* u. \tag{26}$$

After grouping we get

Reformulation of dynamic equations (2/3)

$$M_\xi \ddot{\xi}_1 + C_\xi \dot{\xi}_1 + D_\xi = B_\xi u \qquad (27)$$

where

$$\begin{aligned} M_\xi &= R^T M^* R \\ C_\xi &= R^T \left(M^* \dot{R} + C^* R \right) \\ D_\xi &= R^T D^* \\ B_\xi &= R^T B^* \end{aligned}$$

The $M_\xi(q)$ matrix is inertia matrix expressed in linearised coordinates. Dynamics expressed in linearised coordinates has

Reformulation of dynamic equations (3/3)

following general form

$$M_\xi \ddot{\xi}_1 + C_\xi \dot{\xi}_1 + D_\xi = B_\xi u \tag{28}$$

Property of the system

The system expressed as (28) has the same structure as a manipulator.

Symmetry of M_ξ matrix (1/1)

It is required of the inertia matrix M_ξ to be symmetric because control algorithms require this property.

How to check if J is symmetric?

$$M_\xi \stackrel{?}{=} M_\xi^T \tag{29}$$

$$M_\xi^T = (R^T M^* R)^T = R^T {M^*}^T R = R^T M^* R = M_\xi. \tag{30}$$

Because inertia matrix M^* is symmetric itself ($M^* = {M^*}^T$).

Differentiation of ξ_2 (1/2)

$$\xi_2 = k(q) \tag{31}$$

$$\dot{\xi}_2 = \frac{\partial k}{\partial q}\frac{dq}{dt} = \frac{\partial k}{\partial q}\dot{q} = \frac{\partial k}{\partial q}G\eta = S(q)\dot{\xi}_1 \tag{32}$$

where

$$S(q) = \frac{\partial k}{\partial q}GR \tag{33}$$

Therefore the following equation

$$\dot{\xi}_2 = S(q)\dot{\xi}_1. \tag{34}$$

The above equation is similar to the "kinematics".

Differentiation of ξ_2 (2/2)

In particular ξ_2 can be used to calculate signals (position as well as orientation)

Let us assume that $k(q)$ is given as

$$k(q) = \begin{pmatrix} x \\ y \\ \theta \\ \Phi_1 \\ \Phi_2 \end{pmatrix} = q. \tag{35}$$

Thus,

$$\dot{q} = \dot{\xi}_2 = \frac{\partial k}{\partial q} GR\dot{\xi}_1 = I_{5\times 5} GR\dot{\xi}_1 = GR\dot{\xi}_1. \tag{36}$$

Control low (1/3)

The dynamics expressed in linearised coordinates

$$M_\xi \ddot{\xi}_1 + C_\xi \dot{\xi}_1 + D_\xi = B_\xi u \tag{37}$$

can be expressed as an affine system

$$\begin{cases} \ddot{\xi}_1 & = F_\xi + G_\xi u \\ F_\xi & = -M_\xi^{-1} C_\xi \dot{\xi}_1 - M_\xi^{-1} D_\xi \\ G_\xi & = M_\xi^{-1} B_\xi \end{cases} . \tag{38}$$

Let us consider following control law

$$u = G_\xi^{-1} \left(v - F_\xi \right) . \tag{39}$$

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Control low (2/3)

Full knowledge of the model

The control law (39) assumes that the full knowledge about control object is known.

v is a new input to the system. After injecting (39) into the affine system (38) we get a closed-loop system in the form of a double integrator.

$$\ddot{\xi}_1 = v. \tag{40}$$

Control low (3/3)

To track a desired trajectory in linearised coordinates ξ_1 we propose a PD controller with correction

$$v = \ddot{\xi}_{1d} - K_p e_\xi - K_d \dot{e}_\xi. \tag{41}$$

The errors are defined as $e_\xi = \xi_1 - \xi_{1d}$ and $\dot{e}_\xi = \dot{\xi}_1 - \dot{\xi}_{1d}$. Matrices K_p and K_d are symmetric and positively defined.

Properties of a linearised system

Model expressed in linearised coordinates has following properties:

1. Only the position is linearised.
2. The orientation cannot be linearised.

Eccentric movement

The eccentric movement can be observed because tracking is only done for position.

Unicycle (1/2)

Let us consider an unicycle platform – (2,0) platform. The kinematics is described as

$$\begin{aligned} \dot{x} &= v\cos\theta, \\ \dot{y} &= v\sin\theta, \\ \dot{\theta} &= \omega. \end{aligned}$$

or equivalently

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \dot{q} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} v \\ \omega \end{pmatrix} = G \begin{pmatrix} v \\ \omega \end{pmatrix} \qquad (42)$$

Let us choose linearising functions $h(g)$ as

Unicycle (2/2)

$$h = \begin{pmatrix} h_1(q) \\ h_2(q) \end{pmatrix} = \begin{pmatrix} x + e\cos(\theta+\delta) \\ y + e\sin(\theta+\delta) \end{pmatrix} \tag{43}$$

$$R^{-1} = \frac{\partial h}{\partial q} G = \begin{bmatrix} 1 & 0 & -e\sin(\theta+\delta) \\ 0 & 1 & e\cos(\theta+\delta) \end{bmatrix} \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}$$

$$R^{-1} = \begin{bmatrix} \cos\theta & -e\sin(\theta+\delta) \\ \sin\theta & e\cos(\theta+\delta) \end{bmatrix} \tag{44}$$

Regularity condition

$$
\begin{aligned}
detR^{-1} &= e\cos\theta\cos(\theta+\delta) + e\sin\theta\sin(\theta+\delta) \\
&= e(\cos\theta\cos(\theta+\delta) + \sin\theta\sin(\theta+\delta)) \\
&= e\cos(-\delta) = e\cos\delta
\end{aligned}
$$

The condition is fulfilled when

$$detR^{-1} = e\cos\delta \neq 0,$$

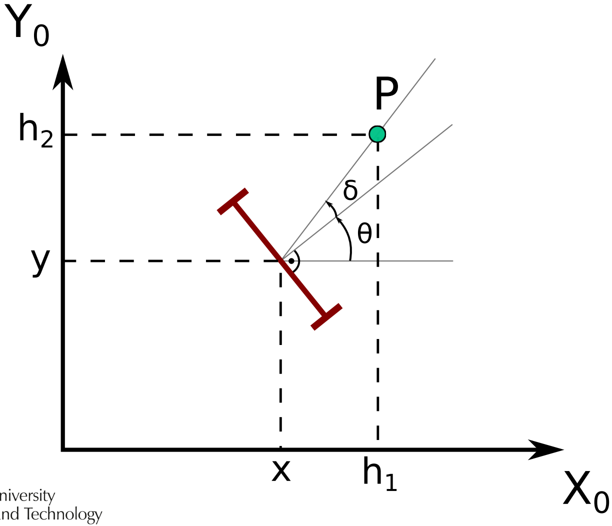
thus, following two conditions should be met

$$e \neq 0 \wedge \delta \neq \frac{\pi}{2} + k\pi. \tag{45}$$

Guidance point P

Considering above the point P cannot lay on the axle.

Geometric interpretation



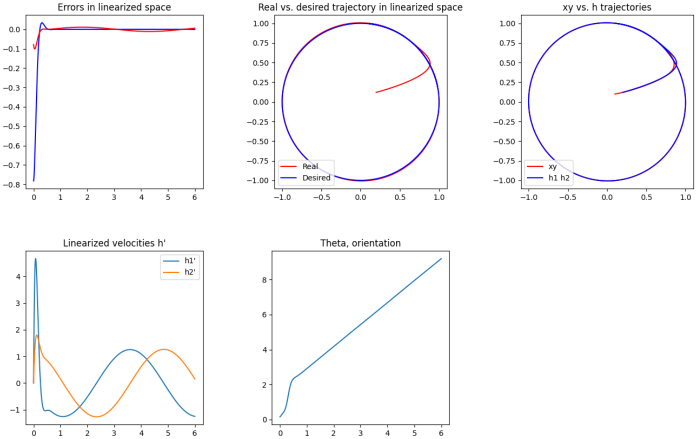
Control problem statement

The objective is to track a desired trajectory ξ_{1d}. The trajectory is given as

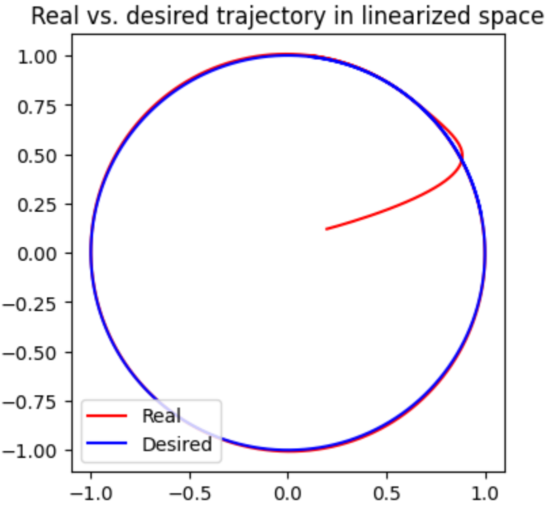
$$\begin{cases} h_{d1} &= A\cos(\omega t + \Delta) \\ h_{d2} &= A\sin(\omega t + \Delta) \end{cases} \tag{46}$$

where $A = 1$, $\omega = \frac{2}{5}\pi$ and $\Delta = 0.2$.
Controller gains are given as $K_p = 200$ and $K_d = 20$.
Guidance point is defined by $e = 0.1$ and $\delta = 0.05$.

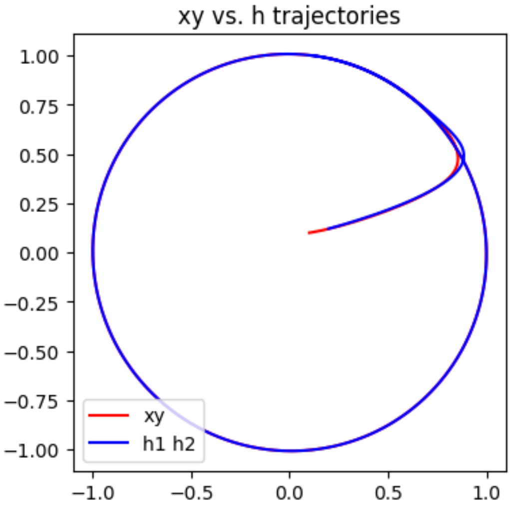
Overview


Errors in linearized space
Real vs. desired trajectory in linearized space
Real
Desired
xy vs. h trajectories
xy
h1 h2
Linearized velocities h'
h1'
h2'
Theta, orientation

Trajectory tracking



Unicycle's centre vs. guidance point



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Error convergence

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Quiz (1/1)

Calculate group number as the rest from dividing the Student ID number by 4.

Example

Student ID number is 123456, thus the group is 0.
Take last 2 digits from Student ID number (56) and calculate the rest from dividing by 4 (56 % 4 = 0).

Write down your name, Student ID number and group.

Literature (1/1)

C. C. de Wit, B. Siciliano, and G. Bastin.
Theory of Robot Control.
Springer-Verlag London, 1996.