Advanced Robot Control

Dynamic linearisation

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Outline

Dynamic linearisation is based on similar concept to static linearisation. A set of $h(q)$ functions is selected that cause regularity condition to **fail**.

The state vector is extended to accommodate for a new "dynamic" variable.

Finally, the dynamic decoupling matrix is calculated. The obtained matrix needs to be non-singular, thus providing additional constraints on the system.

$h(q)$ linearisation functions (1/2)

Let us select following linearising functions

$$\begin{aligned} h_1 &= x + e\cos(\theta + \delta), \\ h_2 &= y + e\sin(\theta + \delta). \end{aligned} \quad (1)$$

Now, let us consider a condition for which the regularity condition was not satisfied, thus $\delta = \frac{\pi}{2}$. Then

$$h_1 = x + e\cos(\theta + \frac{\pi}{2}) = x - e\sin(\theta), \quad (2)$$

$$h_2 = y + e\sin(\theta + \frac{\pi}{2}) = y + e\cos(\theta). \quad (3)$$

$h(q)$ linearisation functions (2/2)

The guidance point that lies on the unicycle axle was singular from perspective of static linearisation. However, for dynamic linearisation this is required condition.

Unicycle kinematics (1/1)

Let us consider kinematic of unicycle in following shape

$$\begin{cases} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega \end{cases} \tag{4}$$

New input to the system (1/2)

Inserting kinematics (4) into derivative of linearising functions (3) yields

$$\dot{h}_1 = \dot{x} - e\cos\theta\dot{\theta} = v\cos\theta - e\cos\theta\omega = \cos\theta(v - e\omega) \quad (5)$$

$$\dot{h}_2 = \dot{y} - e\sin\theta\dot{\theta} = v\sin\theta - e\sin\theta\omega = \sin\theta(v - e\omega) \quad (6)$$

Let us consider

$$\begin{cases} \chi_1 & = v - e\omega \\ \dot{\chi}_1 & = w_1 \end{cases} \quad (7)$$

w_1 is the new input to the linearised system.

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New input to the system (2/2)

Then derivatives of the linearising functions hold following form

$$\begin{cases} \dot{h}_1 &= \chi_1 \cos\theta \\ \dot{h}_2 &= \chi_1 \sin\theta \end{cases} \tag{8}$$

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Extending state vector

Initially, the state vector was composed of position and orientation

$$q = \begin{bmatrix} x & y & \theta \end{bmatrix}^T. \tag{9}$$

Extended state vector holds additional variable χ_1

$$q_e = \begin{bmatrix} x & y & \theta & \chi_1 \end{bmatrix}^T. \tag{10}$$

Dynamic decoupling matrix (1/2)

$$\ddot{h}_1 = \dot{\chi}_1 \cos\theta + \chi_1(-\sin\theta)\dot{\theta} = \dot{\chi}_1 \cos\theta - \omega\chi_1 \sin\theta \tag{11}$$

$$\ddot{h}_2 = \dot{\chi}_1 \sin\theta + \chi_1 \cos\theta\dot{\theta} = \dot{\chi}_1 \sin\theta + \omega\chi_1 \cos\theta \tag{12}$$

Let $w_2 = \omega$ and since $w_1 = \dot{\chi}_1$ then above can be rewritten in matrix form

$$\begin{pmatrix} \ddot{h}_1 \\ \ddot{h}_2 \end{pmatrix} = \begin{bmatrix} \cos\theta & -\chi_1 \sin\theta \\ \sin\theta & \chi_1 \cos\theta \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{13}$$

Dynamic decoupling matrix (2/2)

Further, it can be written as

$$\ddot{h} = K_d(q)w, \tag{14}$$

where K_d is known as dynamic decoupling matrix.
Similarly to the regularity condition for static linearisation, it is required for K_d not to be a singular matrix and invertible, thus

$$\det K_d \neq 0. \tag{15}$$

Constraints (1/2)

For given linearising functions (1) the dynamic decoupling matrix has following shape

$$K_d = \begin{bmatrix} \cos\theta & -\chi_1 \sin\theta \\ \sin\theta & \chi_1 \cos\theta \end{bmatrix}. \tag{16}$$

Calculation of the determinant of above matrix yields

$$\det K_d = \chi_1 \cos^2\theta + \chi_1 \sin^2\theta = \chi_1 \left(\cos^2\theta + \sin^2\theta\right) = \chi_1. \tag{17}$$

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Constraints (2/2)

If we want to control dynamically linearised system

$$\ddot{h} = K_d(q)u \tag{18}$$

then

$$\chi_1 \neq 0. \tag{19}$$

Equivalently

$$v - e\omega \neq 0. \tag{20}$$

The system must be constantly on the move.
The dynamic linearisation is a good choice for trajectory tracking problems.

Diffeomorphism

If a diffeomorphism $f : q_e \to \Phi$ exists between extended state q_e and linear variables Φ,

$$q = \begin{bmatrix} x & y & \theta & \chi_1 \end{bmatrix}^T, \tag{21}$$

$$\Phi = \begin{bmatrix} h_1 & h_2 & \dot{h}_1 & \dot{h}_2 \end{bmatrix}^T m \tag{22}$$

then there is no problem with tracking entire posture including position and **orientation** which was not possible with static linearisation.

Dynamic linearisation allows for full linearisation of state space including orientation.

Control low (1/2)

Given system

$$\ddot{h} = K_d(q)u \tag{23}$$

a following control low can be proposed

$$u = K_d^{-1}(\ddot{h}_d - K_1\dot{e}_h - K_0 e_h), \tag{24}$$

where K_0 and K_1 are positively defined matrices, and errors are $e_h = h - h_d$ and $\dot{e}_h = \dot{h} - \dot{h}_d$.

Control low (2/2)

Let us close the control loop by injecting control low (24) into our system (23).

$$\ddot{h} = K_d(q)K_d^{-1}(\ddot{h}_d - K_1\dot{e}_h - K_0 e_h). \tag{25}$$

$$\ddot{h} = \ddot{h}_d - K_1\dot{e}_h - K_0 e_h. \tag{26}$$

$$\ddot{e}_h + K_1\dot{e}_h + K_0 e_h = 0. \tag{27}$$

Using Laplace transform

$$\mathscr{L}\{\ddot{e}_h + K_1\dot{e}_h + K_0 e_h\} = E_h(s)\left(s^2 + K_1 s + K_0\right). \tag{28}$$

Based on Hurwitz criteria, if $K_0, K_1 > 0$ then (28) is Hurwitz polynomial, thus the system is exponentially stable.

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Real unicyle control signals (1/4)

Given following control law

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = u = K_d^{-1}(\ddot{h}_d - K_1\dot{e}_h - K_0 e_h), \tag{29}$$

we have

$$u = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \dot{\chi}_1 \\ \omega \end{pmatrix} = \begin{pmatrix} \dot{v} - e\dot{\omega} \\ \omega \end{pmatrix}. \tag{30}$$

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Real unicyle control signals (2/4)

If $\dot{\chi}_1 = \dot{v} - e\dot{\omega}$ then

$$\dot{\chi}_1 = \dot{v} - e\dot{\omega} \tag{31}$$

. Above is a concrete function calculated based on (29).

$$\dot{v} = \dot{\chi}_1 + e\dot{\omega} \tag{32}$$

and then

$$\dot{v} = w_1 + e\frac{d}{dt}w_2. \tag{33}$$

To calculate v integration is required.

$$v = \int \left(w_1 + e\frac{d}{dt}w_2 \right), \tag{34}$$

$$v = \int w_1 + ew_2. \tag{35}$$

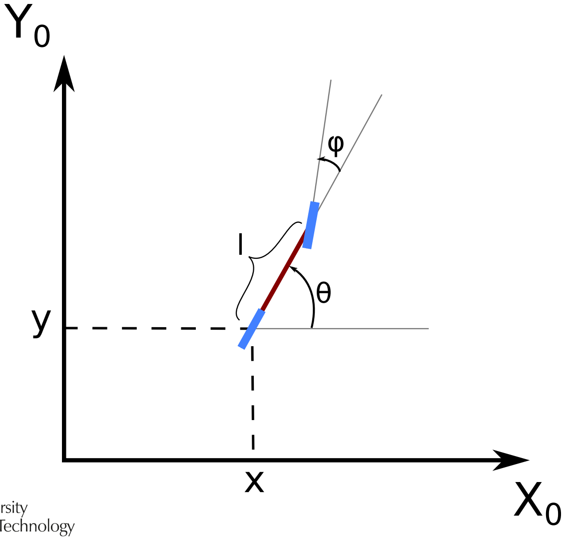
Real unicyle control signals (3/4)

Integration of w_1 is required to calculate real control signals. It also introduces a **delay** into the system.

Real unicyle control signals (4/4)

$\omega = w_2$ is also a concrete function calculated based on control law (29).

Kinematic vehicle (1/2)



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Kinematic vehicle (2/2)

Let us consider kinematic for the object

$$\begin{cases} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \frac{v}{l}\tan\phi \\ \dot{\phi} &= \omega \end{cases}, \tag{36}$$

where v is linear velocity, l is wheel base, while ϕ is attack angle of the steering wheel and ω is angular velocity of the rotating steering wheel.

$h(q)$ linearisation functions (1/6)

Let us consider straightforward linearisation functions

$$h_1 = x$$
$$h_2 = y. \tag{37}$$

The first derivatives are directly taken from kinematics.

$$\dot{h}_1 = v\cos\theta \tag{38}$$
$$\dot{h}_2 = v\sin\theta \tag{39}$$

Further differentiation yields

$h(q)$ linearisation functions (2/6)

$$\begin{aligned}
\ddot{h}_1 &= \dot{v}\cos\theta - v\sin\theta\dot{\theta}\\
\ddot{h}_1 &= \dot{v}\cos\theta - v\sin\theta v\tan\phi\\
\ddot{h}_1 &= \dot{v}\cos\theta - \sin\theta\tan\phi v^2
\end{aligned} \tag{40}$$

and for $\ddot{h}_2$

$$\begin{aligned}
\ddot{h}_2 &= \dot{v}\sin\theta + v\cos\theta\dot{\theta}\\
\ddot{h}_2 &= \dot{v}\sin\theta + v\cos\theta v\tan\phi\\
\ddot{h}_2 &= \dot{v}\sin\theta + \cos\theta\tan\phi v^2.
\end{aligned} \tag{41}$$

$h(q)$ linearisation functions (3/6)

Assuming that $\dot{v} = z$ is a new input to the system we have

$$\ddot{h}_1 = z\cos\theta - \sin\theta\tan\phi v^2, \tag{42}$$

$$\ddot{h}_2 = z\sin\theta + \cos\theta\tan\phi v^2. \tag{43}$$

Further differentiation reveals relationship between new control inputs and state.

$$\dddot{h}_1 = \dot{z}\cos\theta - z\sin\theta\dot{\theta} - \cos\theta\dot{\theta}\tan\phi v^2 - \sin\theta\frac{1}{\cos^2\phi}\dot{\phi}v^2 - \sin\theta\tan\phi 2v\dot{v} \tag{44}$$

$h(q)$ linearisation functions (4/6)

Let $\dot{v} = z$ and $\dot{z} = w_1$, while $\dot{\phi} = \omega$ then

$$\dddot{h}_1 = w_1 \cos\theta - z \sin\theta\dot{\theta} \\ - \cos\theta\dot{\theta}\tan\phi v^2 - \sin\theta\frac{1}{\cos^2\phi}\omega v^2 - \sin\theta\tan\phi 2vz \quad (45)$$

Also $\dot{\theta} = v\tan\phi$, therefore above can be further rewritten and grouped accordingly to control input

$$\dddot{h}_1 = \cos\theta w_1 - \sin\theta\frac{1}{\cos^2\phi}v^2\omega \\ - \cos\theta\tan^2\phi v^3 - 3zv\sin\theta v\tan\phi \quad (46)$$

Similarly for $\dddot{h}_2$

$h(q)$ linearisation functions (5/6)

$$\dddot{h}_2 = \dot{z}\sin\theta + z\cos\theta\dot{\theta} - \sin\theta\dot{\theta}\tan\phi v^2 + \cos\theta\frac{1}{\cos^2\phi}\dot{\phi}v^2 + \cos\theta\tan\phi 2v\dot{v} \quad (47)$$

Let $\dot{v} = z$ and $\dot{z} = w_1$, while $\dot{\phi} = \omega$ then

$$\dddot{h}_2 = w_1\sin\theta + z\cos\theta\dot{\theta} - \sin\theta\dot{\theta}\tan\phi v^2 + \cos\theta\frac{1}{\cos^2\phi}\omega v^2 + \cos\theta\tan\phi 2vz \quad (48)$$

Let us group above and use $\dot{\theta} = v\tan\phi$ relation

$$\dddot{h}_2 = \sin\theta w_1 + \cos\theta\frac{1}{\cos^2\phi}v^2\omega - \sin\theta\tan^2\phi v^3 + 3\cos\theta\tan\phi vz \quad (49)$$

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$h(q)$ linearisation functions (6/6)

Finally, the dynamically linearised system can be written in matrix form.

$$\begin{pmatrix} \dddot{h}_1 \\ \dddot{h}_2 \end{pmatrix} = K_d \begin{pmatrix} w_1 \\ \omega \end{pmatrix} + f, \tag{50}$$

where

$$K_d = \begin{bmatrix} \cos\theta & -\sin\theta \frac{1}{\cos^2\phi} v^2 \\ \sin\theta & \cos\theta \frac{1}{\cos^2\phi} v^2 \end{bmatrix}, \tag{51}$$

$$f = \begin{pmatrix} -\cos\theta \tan^2\phi v^3 - 3vz \sin\theta \tan\phi \\ -\sin\theta \tan^2\phi v^3 + 3vz \cos\theta \tan\phi \end{pmatrix}. \tag{52}$$

Constraints (1/3)

The dynamic decoupling matrix has following shape

$$K_d = \begin{bmatrix} \cos\theta & -\sin\theta \frac{1}{\cos^2\phi} v^2 \\ \sin\theta & \cos\theta \frac{1}{\cos^2\phi} v^2 \end{bmatrix}, \tag{53}$$

Calculation of the determinant of above matrix yields

$$\begin{aligned} \det K_d &= \cos^2\theta \frac{1}{\cos^2\phi} v^2 + \sin^2\theta \frac{v^2}{\cos^2\phi} \\ &= \frac{v^2}{\cos^2\phi}\left(\cos^2\theta + \sin^2\theta\right) = \frac{v^2}{\cos^2\phi}. \end{aligned} \tag{54}$$

Constraints (2/3)

If we want to control dynamically linearised system

$$\dddot{h} = K_d(q)u + f \tag{55}$$

then

$$\frac{v^2}{\cos^2\phi} \neq 0. \tag{56}$$

Equivalently

$$\begin{cases} v \neq 0 \\ \cos^2\phi \neq 0 \to \phi \neq \frac{\pi}{2} + k\pi \end{cases}. \tag{57}$$

Constraints (3/3)

$$v \neq 0. \tag{58}$$

The system must be constantly on the move.
The dynamic linearisation is a good choice for trajectory tracking problems.

$$\phi \neq \frac{\pi}{2} + k\pi. \tag{59}$$

The steering wheel cannot be perpendicular to the main axle.

Control low (1/2)

Now, we have our system in following form

$$\dddot{h} = K_d u + f. \tag{60}$$

Let us propose a control law for (60).

$$u = K_d^{-1}(-f + \dddot{h}_d - K_2 \ddot{e}_h - K_1 \dot{e}_h - K_0 e_h), \tag{61}$$

where K_0, K_1 and K_2 are positively defined matrices, and errors are $e_h = h - h_d$, $\dot{e}_h = \dot{h} - \dot{h}_d$ and $\ddot{e}_h = \ddot{h} - \ddot{h}_d$.
To determine if system is stable we need to apply control law (61) to our system (60).

$$\dddot{h} = K_d(q) K_d^{-1}(-f + \dddot{h}_d - K_2 \ddot{e}_h - K_1 \dot{e}_h - K_0 e_h) + f. \tag{62}$$

$$\dddot{h} = \dddot{h}_d - K_2 \ddot{e}_h - K_1 \dot{e}_h - K_0 e_h. \tag{63}$$

$$\dddot{e}_h + K_2 \ddot{e}_h + K_1 \dot{e}_h + K_0 e_h = 0. \tag{64}$$

Control low (2/2)

Using Laplace transform

$$\mathscr{L}\{\dddot{e}_h + K_2\ddot{e}_h + K_1\dot{e}_h + K_0 e_h\} = E_h(s)\left(s^3 + K_2 s^2 + K_1 s + K_0\right). \tag{65}$$

Based on Hurwitz criteria, if $K_0, K_1, K_2 > 0$ then (65) is Hurwitz polynomial, thus the system is exponentially stable.

Real kinematic vehicle control signals (1/3)

Given following control law

$$\begin{pmatrix} w_1 \\ \omega \end{pmatrix} = u = K_d^{-1}(\dddot{h}_d - K_2\ddot{e}_h - K_1\dot{e}_h - K_0 e_h), \tag{66}$$

we have

$$u = \begin{pmatrix} w_1 \\ \omega \end{pmatrix} = \begin{pmatrix} \dot{z} \\ \omega \end{pmatrix} = \begin{pmatrix} \ddot{v} \\ \omega \end{pmatrix}. \tag{67}$$

Real kinematic vehicle control signals (2/3)

$$\ddot{v} = w_1 . \tag{68}$$

Above is a concrete function calculated based on (66). To calculate v integration is required.

$$v = \int \int w_1 , \tag{69}$$

Integration of w_1 is required to calculate real control signals. It also introduces a **delay** into the system.

Real kinematic vehicle control signals (3/3)

$\omega = w_2$ is also a concrete function calculated based on control law (66).

Literature (1/1)

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